Practice Midterm Exam 2

This exam is closed-book and closed-computer. You may have a double-sided, 8.5" × 11" sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices during the course of this exam without prior authorization from the course staff. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 32 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

Question	Points
(1) Set Theory	/ 8
(2) Mathematical Logic	/ 8
(3) Proofwriting I	/ 8
(4) Proofwriting II	/ 8
	/ 32

Good luck!

Problem Two: Mathematical Logic   (8 Points)

(CS103 Midterm, Fall 2016)

i. **(6 Points)** Suppose that *A* and *B* are sets where $A \subseteq B$. Below is a series of six statements about *A* and *B*. For each statement, decide whether it's always true, always false, or depends on the choice of *A* and *B*. If you choose that last option, provide one example of an *A* and *B* where the statement is true and one example of an *A* and *B* where the statement is false. (Remember that, in your examples, you need to have $A \subseteq B$.)

$$A \neq B$$

☐ Always True   ☐ Never True   ☐ It depends. For example…

… it's true if *A* = _____ and *B* = _____

… it's false if *A* = _____ and *B* = _____

$$A \in B$$

☐ Always True   ☐ Never True   ☐ It depends. For example…

… it's true if *A* = _____ and *B* = _____

… it's false if *A* = _____ and *B* = _____

$$A \in \wp(B)$$

☐ Always True   ☐ Never True   ☐ It depends. For example…

… it's true if *A* = _____ and *B* = _____

… it's false if *A* = _____ and *B* = _____

$$B \in \wp(A)$$

☐ Always True   ☐ Never True   ☐ It depends. For example…

… it's true if *A* = _____ and *B* = _____

… it's false if *A* = _____ and *B* = _____

$$A \subseteq \wp(B)$$

☐ Always True   ☐ Never True   ☐ It depends. For example…

… it's true if *A* = _____ and *B* = _____

… it's false if *A* = _____ and *B* = _____

$$B \subseteq \wp(A)$$

☐ Always True   ☐ Never True   ☐ It depends. For example…

… it's true if *A* = _____ and *B* = _____

… it's false if *A* = _____ and *B* = _____

(CS103 Midterm, Fall 2016)

ii. **(2 Points)** Suppose that S and T are sets. Which of the following first-order logic statements are translations of the statement "S is not a subset of T?" Check all that apply.

$\square\ \forall x.\ (x \in S \rightarrow x \notin T)$

$\square\ \forall x.\ (x \in S \land x \notin T)$

$\square\ \exists x.\ (x \in S \rightarrow x \notin T)$

$\square\ \exists x.\ (x \in S \land x \notin T)$

Problem Two: Mathematical Logic   (8 Points)

(CS103 Midterm, Fall 2016)

When we discussed first-order logic, we spent a decent amount of time talking about how to trans- late statements from English into first-order logic. You practiced this skill on Problem Set Two. In this prob- lem, we'd like you to show us what you've learned about the art of first-order translation.

In Nikolai Gogol's story "The Nose," the protagonist Major Kovalyov wakes up and finds that his nose is missing. Later on, he sees his nose walking around in plain sight. Everyone sees the nose, but only Kova- lyov is perplexed by this. (There's more to the story than that, of course, and after this exam, I highly rec- ommend that you read it – it's a great social commentary.) In the meantime, though, we'd like you to translate the gist of the plot of the story into first-order logic.

    i. **(5 Points)** Given the constant symbols

        Kovalyov, which represents Kovalyov the protagonist, and
        TheNose, which represents Kovalyov's nose,

    and the predicates

        Person(p), which says that p is a person;
        Sees(a, b), which says that a sees b; and
        IsPerplexed(x), which says that x is perplexed,

    write a formula in first-order logic that says "all people see the nose, but Kovalyov is the only per- son who's perplexed."

(CS103 Midterm, Fall 2016)

We've talked a lot about negating and simplifying statements in first-order logic, a useful skill with applications to proofs by contradiction and contrapositive. You practiced this on Problem Set Two, and in this question we'd like you to demonstrate what you've learned.

ii. **(3 Points)** Consider the following statement in first-order logic:

$\exists x.\ (Person(x)\ \wedge$
  $\forall y.\ (Person(y) \wedge (\ \forall z.\ (Kitten(z) \rightarrow Loves(y, z))\ ) \rightarrow$
    $(Loves(x, y) \leftrightarrow Loves(y, x))$
  $)$
$)$

Give a statement in first-order logic that is the negation of this statement. As in Problem Set Two, your final formula must not have any negations in it, except for direct negations of predicates. Do not introduce any new predicates, functions, or constants.

Problem Three: Proofwriting I   (8 Points)

*(An old problem set question from the Days of Yore)*

Imagine an infinitely long sequence of squares, such as below:

One of these squares contains a frog, and another square contains a fly:

For simplicity, let's number all of the (infinitely many) squares by assigning each an integer. We'll say that the frog starts in position 0, and will assign positive integers to the squares to the right of the frog and negative numbers to the squares to the left of the frog. For example:
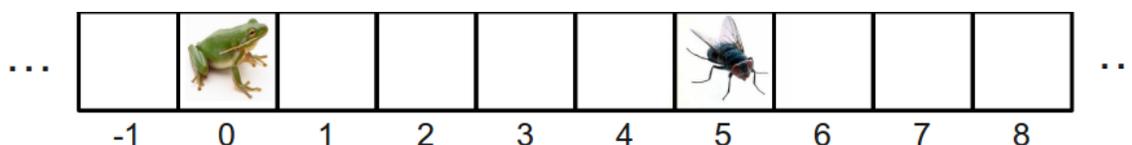
Now this frog is a very special kind of frog called a *quantum frog*. The quantum frog can hop across the squares forward and backward, but can only make jumps of two different lengths: 3 and 7. For example, to get to square five to eat the fly, the frog might might jump forward seven squares to square 7, forward seven squares again to square 14, then back three squares three times to squares 11, 8, and (finally) 5.

Prove that, starting at position 0, the quantum frog can move to any square using only jumps of length 3 and 7.

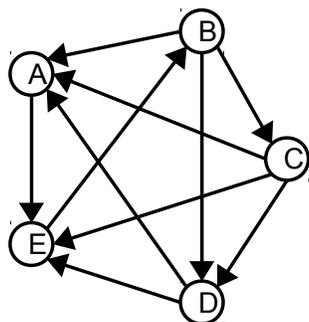*(Extra space for your answer to Problem Three, if you need it.)*

Problem Four: Proofwriting II   (8 Points)

(CS103 Midterm, Spring 2015)

On Problem Set Two, you explored tournaments, contests between groups of *n* players. This problem explores some other properties of tournaments.

Let's begin by refreshing some definitions. A ***tournament*** is a contest among *n* players. Each player plays a game against each other player, and either wins or loses the game (let's assume that there are no draws). We can visually represent a tournament by drawing a circle for each player and drawing arrows between pairs of players to indicate who won each game. For example, in the tournament to the left, player *A* beat player *E*, but lost to players *B*, *C*, and *D*.

A ***tournament winner*** is a player in a tournament who, for each other player, either won her game against that player, or won a game against a player who in turn won his game against that player (or both). In the tournament to the left, players *B*, *C*, and *E* are tournament winners. As you proved on Problem Set Two, every tournament with at least one player will have at least one tournament winner.

Let's introduce a new definition. If *T* is a tournament, a ***subtournament*** of *T* is a tournament *T'* formed by picking a subset of the players in *T* and considering just the games those players played against each other. (Note that, just like any set is a subset of itself, any tournament is a subtournament of itself).

Let *T* be an arbitrary tournament and let *p* be an arbitrary player in the tournament. Prove that if anyone beat *p*, then at least one of the players who beat *p* was a tournament winner. As a hint, split the tournament *T* into three pieces: a subtournament consisting of the players who won against *p*, a subtournament consisting of the players who lost to *p*, and player *p* herself. Look at those subtournaments and see if there's anything interesting you can say about them.

(Extra space for your answer to Problem Four, if you need it.)